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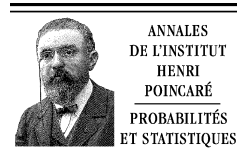
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# Branching Brownian motion with an inhomogeneous breeding potential

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**Abstract.** This article concerns branching Brownian motion (BBM) with dyadic branching at rate  $\beta|y|^p$  for a particle with spatial position  $y \in \mathbb{R}$ , where  $\beta > 0$ . It is known that for  $p > 2$  the number of particles blows up almost surely in finite time, while for  $p = 2$  the expected number of particles alive blows up in finite time, although the number of particles alive remains finite almost surely, for all time. We define the right-most particle,  $R_t$ , to be the supremum of the spatial positions of the particles alive at time  $t$  and study the asymptotics of  $R_t$  as  $t \rightarrow \infty$ . In the case of constant breeding at rate  $\beta$  the linear asymptotic for  $R_t$  is long established. Here, we find asymptotic results for  $R_t$  in the case  $p \in (0, 2]$ . In contrast to the linear asymptotic in standard BBM we find polynomial asymptotics of arbitrarily high order as  $p \uparrow 2$ , and a non-trivial limit for  $\ln R_t$  when  $p = 2$ . Our proofs rest on the analysis of certain additive martingales, and related spine changes of measure.

**Résumé.** Cet article concerne un mouvement brownien branchant (BBM) en deux particules avec un taux  $\beta|y|^p$  pour une particule située en  $y \in \mathbb{R}$ , avec une constante  $\beta > 0$ . Il est connu que pour  $p > 2$ , le nombre de particules explose presque sûrement en temps fini, alors que pour  $p = 2$  le nombre de particules explose en moyenne en temps fini bien qu'il reste fini presque sûrement à tout moment. Nous définissons la particule la plus à droite  $R_t$  comme le supremum des positions spatiales des particules vivant à l'instant  $t$  et étudions les asymptotiques de  $R_t$  quand  $t$  tend vers l'infini. Dans le cas d'une reproduction à taux constant  $\beta$ , l'asymptotique linéaire de  $R_t$  est bien connue. Ici, nous trouvons des résultats asymptotiques pour  $R_t$  dans le cas où  $p \in (0, 2]$ . Contrastant avec les asymptotiques linéaires du BBM standard, nous trouvons des asymptotiques polynomiales de degré arbitrairement grand quand  $p$  croît vers 2, et une limite non triviale pour  $\ln R_t$  quand  $p = 2$ . Nos preuves s'appuient sur certaines martingales positives et des changements de mesures.

MSC: 60J80

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## 1. Introduction and notation

We consider a branching Brownian motion with an inhomogeneous breeding potential. Each particle diffuses as a driftless Brownian motion and splits into two particles at rate  $\beta|y|^p$ , where  $\beta > 0$ ,  $p \in [0, 2]$  and  $y \in \mathbb{R}$  is the particle's spatial position. The set of particles alive at time  $t$  is  $N_t$ , and then, for each  $u \in N_t$ ,  $Y_u(t)$  is the spatial position of particle  $u$  at time  $t$ . In the sequel we refer to this process as a  $(\beta|y|^p; \mathbb{R})$ -BBM, with probabilities  $\{P^x : x \in \mathbb{R}\}$ , where  $P^x$  is the law of the process started from a single particle at the point  $x \in \mathbb{R}$ .  $E^x$  will be the expectation operator for  $P^x$ .

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It is known from Itô and McKean [8], pp. 200–211, that quadratic breeding is a critical rate for population explosions. If the breeding rate were instead  $\beta|y|^p$  for  $p > 2$ , the population would almost surely explode in a finite time. However, for the  $(\beta y^2; \mathbb{R})$ -BBM the *expected* number of particles blows up in a finite time, but the total number of particles alive remains finite almost surely, for all time. The fact that expectations for this process are not well behaved adds to the difficulty of its study.

We define  $R_t := \sup_{u \in N_t} Y_u(t)$  to be the right-most particle in a  $(\beta|y|^p; \mathbb{R})$ -BBM. It is very well known that for  $p = 0$ , i.e., constant breeding at rate  $\beta$ , the right-most particle satisfies  $\lim_{t \rightarrow \infty} t^{-1} R_t = \sqrt{2\beta}$ , almost surely. More precise limit results concerning sub-linear correction terms in this asymptotic have also been found, see Bramson [2] and [3], for example.

In this paper we find the leading order term in the right-most particle asymptotic for all  $p \in [0, 2]$ ; the method of proof in all cases is almost identical, but for ease of exposition we separate the case of the quadratic breeding potential.

**Theorem 1.** (a)  $(\beta|y|^p; \mathbb{R})$ -BBM,  $p \in [0, 2)$ .

$$\lim_{t \rightarrow \infty} \frac{R_t}{t^{\hat{b}}} = \hat{a}$$

$P^x$ -almost surely, where

$$\hat{a} := \left( \frac{\beta}{2} (2-p)^2 \right)^{1/(2-p)} \quad \text{and} \quad \hat{b} := \frac{2}{2-p}.$$

(b)  $(\beta y^2; \mathbb{R})$ -BBM.

$$\lim_{t \rightarrow \infty} \frac{\ln R_t}{t} = \sqrt{2\beta}$$

$P^x$ -almost surely.

Thus we see polynomial spatial growth in the  $(\beta|y|^p; \mathbb{R})$ -BBM of arbitrarily high power as  $p \uparrow 2$ , and the right-most particle in the  $(\beta y^2; \mathbb{R})$ -BBM has spatial displacement that is, asymptotically, of exponential order; contrast this with the linear spread of standard BBM.

To prove Theorem 1 we use martingale arguments – in particular an additive martingale and a spine change of measure. Section 2 introduces the spine ideas and details the changes of measure we use, as well as defining the additive martingale which turns out to be the key tool in the proof of these results. In Section 3 we study the convergence properties of this martingale, and this allows us to prove the main results in Section 4.

## 2. Spines, measures and martingales

In this section we give the background results and notation required for the spine set-up. We give a more general formulation here than is required for the rest of the paper. Spine constructions were introduced by Chauvin and Rouault [4] in the context of standard branching Brownian motion, and since the series of papers Lyons et al. [12], Lyons [11] and Kurtz et al. [9] have found wide application in the theory of branching processes. For some more recent spine ideas applied to branching diffusions, see Kyprianou [10], or Engländer and Kyprianou [6], for example. For more details on the spine set-up we use below, see Hardy and Harris [7].

Recall that the  $(\beta|y|^p; \mathbb{R})$ -BBM, given by the point process  $\mathbb{Y}_t := \{Y_u(t) : u \in N_t\}$ , where  $N_t$  is the set of particles alive at time  $t$ , has associated probability measures  $P^x$  with natural filtration  $\{\mathcal{F}_t\}_{t \geq 0}$ . All particles are labelled according to the Ulam-Harris convention so that, for example, 412 is ‘the individual being the second child of the first child of the fourth child of the initial ancestor,  $\emptyset$ .’ For two labels  $u, v$  the notation  $v < u$  means that  $v$  is an ancestor of  $u$ , and  $|u|$  denotes the generation of  $u$ .

A *spine*,  $\xi$ , is a distinguished infinite line of descent from the initial ancestor and is given by  $\xi = \{\emptyset, \xi_1, \xi_2, \dots\}$ , where  $\xi_n$  is the label of the spine at the  $n$ -th generation and  $u \in \xi$  means that either  $u = \emptyset$  or  $u = \xi_i$  for some  $i \in \mathbb{N}$ . It is natural to think of the spine as a single diffusing particle, and we denote its path by  $\{\xi_t\}_{t \geq 0}$ . Let  $n = \{n_t\}_{t \geq 0}$  be

the counting function for the number of fissions that have occurred along the path of the spine by time  $t$ , with the set of fission times themselves being  $\{S_i\}_{i \in \mathbb{N}}$ . In particular,  $\xi_{n_t}$  is the label of the spine at time  $t$ .

We now define several filtrations that will be very important in our later proofs. The augmented filtration  $\{\tilde{\mathcal{F}}_t\}_{t \geq 0}$  for the  $(\beta|y|^p; \mathbb{R})$ -BBM with a distinguished spine is defined by  $\tilde{\mathcal{F}}_t := \sigma(\mathcal{F}_t, \{\xi_{n_s}\}_{s \leq t})$ . Thus, in addition to the genealogy and paths of the particles alive at time  $t$ , the filtration  $\tilde{\mathcal{F}}_t$  knows which particle is the spine at all times  $s \in [0, t]$ . Also define  $\mathcal{G}_t := \sigma(\{\xi_s\}_{s \leq t})$  and  $\tilde{\mathcal{G}}_t := \sigma(\mathcal{G}_t, \{n_s, \xi_{n_s}\}_{s \leq t})$ . This means that  $\mathcal{G}_t$  knows only the spine's *motion*; besides this,  $\tilde{\mathcal{G}}_t$  also knows the *spine's* genealogy and the *spine's* fission times; neither knows anything about what happens off the spine.

On the filtration  $\tilde{\mathcal{F}}_t$  we will define a measure  $\tilde{P}^x$  that is the law of the branching Brownian motion  $\{\mathbb{Y}_t\}_{t \geq 0}$  with a distinguished spine. By this we mean that the  $(\beta|y|^p; \mathbb{R})$ -BBM may be constructed under  $\tilde{P}^x$  as follows:

- the initial ancestor (the spine process,  $\xi_t$ ) diffuses as a standard Brownian motion, started at  $x$ ;
- the fission times on the spine occur as a Poisson process of instantaneous rate  $\beta|\xi_t|^p$ , which is independent of the spine's motion;
- at each fission time on the spine two particles are produced;
- one of these is chosen uniformly at random to continue the spine, and it repeats stochastically the behaviour of its parent;
- the other particle initiates an independent  $(\beta|y|^p; \mathbb{R})$ -BBM, with law  $P^*$ , started from its birth position.

(Note that  $\tilde{P}^x$  is an extension of the original measure  $P^x$ , with  $P^x = \tilde{P}^x|_{\mathcal{F}_\infty}$ .)

With these augmented measures and filtrations we can define a one-particle martingale that we will use to change the behaviour of the spine. Let  $g \in C^1(0, \infty)$  satisfy  $\int_0^t g'(s)^2 ds < +\infty$  for all  $t > 0$  and define

$$\tilde{M}_g(t) := e^{-\beta \int_0^t |\xi_s|^p ds} 2^{n_t} \times \exp\left(\int_0^t g'(s) d\xi_s - \int_0^t \frac{1}{2} g'(s)^2 ds\right).$$

Observe that  $\tilde{M}_g$  is itself the product of two  $\tilde{P}^x$ -martingales, the first of which increases the breeding rate along the spine, and the second causes the spine to diffuse as an Brownian motion about the path  $g$ . More precisely, we define a measure  $\tilde{\mathbb{Q}}_g^x$  via

$$\frac{d\tilde{\mathbb{Q}}_g^x}{d\tilde{P}^x} \Big|_{\tilde{\mathcal{F}}_t} = \tilde{M}_g(t),$$

and then the  $(\beta|y|^p; \mathbb{R})$ -BBM with a distinguished spine can be re-constructed in law under  $\tilde{\mathbb{Q}}_g^x$  as:

- the initial ancestor (the spine) diffuses as

$$d\xi_t = d\tilde{B}_t + g'(t) dt, \tag{1}$$

where  $\tilde{B}$  is a  $\tilde{\mathbb{Q}}_g^x$ -Brownian motion;

- the fission times on the spine occur as a Poisson process of instantaneous rate  $2\beta|\xi_t|^p$ , which is independent of the spine's motion;
- at each fission time on the spine two particles are produced;
- one of these is chosen uniformly at random to be the spine, and it repeats stochastically the behaviour of its parent;
- the other particle initiates an independent  $(\beta|y|^p; \mathbb{R})$ -BBM, with law  $P^*$ , started from its birth position.

Furthermore, our construction of the spine foundations in terms of filtrations and sub-filtrations allows us to define a measure  $\mathbb{Q}_g^x := \tilde{\mathbb{Q}}_g^x|_{\mathcal{F}_\infty}$ . It can be shown that

$$\frac{d\mathbb{Q}_g^x}{dP^x} \Big|_{\mathcal{F}_t} = M_g(t), \tag{2}$$

where

$$M_g(t) = \sum_{u \in N_t} \exp\left(\int_0^t g'(s) dY_u(s) - \int_0^t \left(\frac{1}{2} g'(s)^2 + \beta|Y_u(s)|^p\right) ds\right)$$

is an additive martingale for the  $(\beta|y|^p; \mathbb{R})$ -BBM, and where, for  $u \in N_t$  and times  $s < t$ ,  $Y_u(s)$  is the position of the unique ancestor of  $u$  that is alive at time  $s$ .

Since  $\mathbb{Q}_g^x$  was defined as a restriction of  $\tilde{\mathbb{Q}}_g^x$  we have that the path-wise constructions of  $\mathbb{Y}$  under  $\mathbb{Q}_g^x$  and  $\tilde{\mathbb{Q}}_g^x$  coincide. It is the martingale  $M_g$  which turns out to be the key tool in proving Theorem 1.

### 3. Convergence properties of $M_g$

For notational convenience we have defined the martingale  $M_g$  for a general path  $g$ , although in this section we will consider only those classes of function  $g$  we need in order to prove Theorems 1.

**Theorem 2.** (a)  $(\beta|y|^p; \mathbb{R})$ -BBM,  $p \in [0, 2)$ . Consider paths  $g(s) = as^b$  for  $a > 0$  and  $b > 1$ .

- (i) If either  $b < \hat{b} = \frac{2}{2-p}$ , or  $b = \hat{b}$  and  $a < \hat{a} = (\frac{\beta}{2}(2-p)^2)^{1/(2-p)}$ , then  $M_g$  is uniformly integrable and  $M_g(\infty) > 0$  almost surely.
- (ii) If either  $b > \hat{b}$ , or  $b = \hat{b}$  and  $a > \hat{a}$ , then  $M_g(\infty) = 0$  almost surely.

(b)  $(\beta y^2; \mathbb{R})$ -BBM. Consider paths  $g(s) = e^{\lambda s}$ , for  $\lambda > 0$ .

- (i) If  $0 < \lambda < \sqrt{2\beta}$ ,  $M_g$  is uniformly integrable and  $M_g(\infty) > 0$  almost surely.
- (ii) If  $\lambda > \sqrt{2\beta}$  then  $M_g(\infty) = 0$  almost surely.

We do not treat the critical martingales, i.e.,  $a = \hat{a}$  and  $b = \hat{b}$ , or  $\lambda = \sqrt{2\beta}$ , as they are not needed to prove Theorem 1; we anticipate, however, that the martingale limits are 0 in these special cases. Essential to our argument is the spine change of measure and the following measure theoretic result, cf. Durrett [5], p. 242, or Athreya [1]. Let  $\mu$  and  $\nu$  be two measures on a probability space  $(\Omega, \{\mathcal{F}_t\}_{t \geq 0}, \mathcal{F}_\infty)$ , related by the Radon–Nikodým derivative

$$\left. \frac{d\mu}{d\nu} \right|_{\mathcal{F}_t} = Z(t),$$

for some  $\nu$ -martingale  $Z$ . Let  $\bar{Z} := \limsup_{t \rightarrow \infty} Z(t)$ , so that  $\bar{Z} = \lim_{t \rightarrow \infty} Z(t)$  almost surely under  $\nu$ , and then

$$\bar{Z} < \infty \quad \mu\text{-a.s.} \quad \Longleftrightarrow \quad \nu(\bar{Z}) = Z(0), \quad (3)$$

$$\bar{Z} = \infty \quad \mu\text{-a.s.} \quad \Longleftrightarrow \quad \bar{Z} = 0 \quad \nu\text{-a.s.} \quad (4)$$

Related spine techniques for martingale convergence go back to Lyons et al. [12].

**Proof of Theorem 2:  $\mathcal{L}^1$  convergence parts.** We will deal initially with the statements telling us when  $M_g$  is uniformly integrable (Theorem 2 parts a(i) and b(i)). The first step is to decompose the martingale  $M_g$  by conditioning on the spine's path,  $\{\xi_t\}_{t \geq 0}$ , and fission times,  $\{S_u: u \in \xi\}$ . By conditioning on  $\tilde{\mathcal{G}}_\infty$  we have

$$\begin{aligned} \tilde{\mathbb{Q}}_g^x(M_g(t) | \tilde{\mathcal{G}}_\infty) &= \sum_{u < \xi_t} \exp\left(\int_0^{S_u} g'(s) d\xi_s - \int_0^{S_u} \left(\frac{1}{2}g'(s)^2 + \beta|\xi_s|^p\right) ds\right) \\ &\quad + \exp\left(\int_0^t g'(s) d\xi_s - \int_0^t \left(\frac{1}{2}g'(s)^2 + \beta|\xi_s|^p\right) ds\right), \end{aligned}$$

and we refer to the two pieces of this decomposition as  $\text{sum}(t)$  and  $\text{spine}(t)$ . We wish to show that the conditional expectation above is  $\tilde{\mathbb{Q}}_g^x$ -almost surely bounded as  $t \rightarrow \infty$ .

From (1) we have

$$\text{spine}(t) = \exp\left(\int_0^t g'(s) d\tilde{B}_s + \int_0^t \left(\frac{1}{2}g'(s)^2 - \beta|\tilde{B}_s + g(s)|^p\right) ds\right),$$

where  $\tilde{B}$  is a  $\tilde{\mathbb{Q}}_g^x$ -Brownian motion started at  $x$ . As  $t \rightarrow \infty$ ,  $\tilde{B}_t/t \rightarrow 0$ ,  $\tilde{\mathbb{Q}}_g^x$ -a.s. and more generally

$$\frac{\int_0^t g'(s) d\tilde{B}_s}{\int_0^t g'(s)^2 ds} \rightarrow 0, \quad \tilde{\mathbb{Q}}_g^x\text{-a.s.}$$

So for any  $\varepsilon, \delta > 0$  there exist random times  $T_\delta, T_\varepsilon < \infty$  such that

$$(1 - \varepsilon)g(s)^p \leq |\tilde{B}_s + g(s)|^p \leq (1 + \varepsilon)g(s)^p \quad \text{for all } s > T_\varepsilon, \quad (5)$$

and

$$-\delta \int_0^t g'(s)^2 ds \leq \int_0^t g'(s) d\tilde{B}_s \leq \delta \int_0^t g'(s)^2 ds \quad \text{for all } t > T_\delta. \quad (6)$$

Hence, for some random, almost-surely finite, constant  $C > 0$  we get the upper bound

$$\text{spine}(t) \leq C \exp\left(\int_{T_\delta \vee T_\varepsilon}^t \left((1 + \delta)\frac{1}{2}g'(s)^2 - \beta(1 - \varepsilon)g(s)^p\right) ds\right) \quad (7)$$

$\tilde{\mathbb{Q}}_g^x$ -almost surely for  $t > T_\delta \vee T_\varepsilon$ . We now consider the cases  $p \in [0, 2)$  and  $p = 2$  separately.

(a) Let  $p \in (0, 2)$ , so that Eq. (7) gives

$$\text{spine}(t) \leq C \exp\left((1 + \delta) \int_{T_\delta \vee T_\varepsilon}^t \left(\frac{1}{2}(ab)^2 s^{2(b-1)} - \frac{1 - \varepsilon}{1 + \delta} \beta a^p s^{pb}\right) ds\right)$$

$\tilde{\mathbb{Q}}_g^x$ -a.s. for  $t > T_\delta \vee T_\varepsilon$ . If  $b < \hat{b} = \frac{2}{2-p}$  then the bound above is  $\tilde{\mathbb{Q}}_g^x$ -a.s. of exponential order  $-ct^{pb+1}$  as  $t \rightarrow \infty$ , for some constant  $c > 0$ . If  $b = \hat{b}$  then the second exponent in the bound above is

$$\left(\frac{2a^2}{(2-p)^2} - a^p \beta \frac{1 - \varepsilon}{1 + \delta}\right) \int_{T_\delta \vee T_\varepsilon}^t s^{2p/(2-p)} ds, \quad (8)$$

and the sign of this expression is negative if and only if

$$a < \left(\left(\frac{1 - \varepsilon}{1 + \delta}\right) \frac{\beta(2-p)^2}{2}\right)^{1/(2-p)}.$$

Since  $\delta, \varepsilon$  can be chosen arbitrarily small we have a decaying upper bound for  $\text{spine}(t)$  in the case  $b = \hat{b}$  and  $0 < a < \hat{a}$ .

(b) If  $p = 2$  then we consider paths  $g(s) = e^{\lambda s}$  and our bound from Eq. (7) is

$$\text{spine}(t) \leq C \exp\left((1 + \delta) \int_{T_\delta \vee T_\varepsilon}^t \left(\frac{\lambda^2}{2} - \beta \frac{1 - \varepsilon}{1 + \delta}\right) e^{2\lambda s} ds\right)$$

$\tilde{\mathbb{Q}}_g^x$ -a.s. for  $t > T_\varepsilon$ . We can choose  $\delta, \varepsilon$  such that this decays for any  $\lambda \in (0, \sqrt{2\beta})$ .

Turning our attention to  $\text{sum}(t)$ , we return to the general case  $p \in [0, 2]$ . From (7) we have, for  $t > T_\delta \vee T_\varepsilon$ ,

$$\begin{aligned} \text{sum}(t) &\leq \sum_{\substack{u < \xi_t \\ S_u \leq T_\delta \vee T_\varepsilon}} \exp\left(\int_0^{S_u} g'(s) d\xi_s - \int_0^{S_u} \left(\frac{1}{2}g'(s)^2 + \beta|\xi_s|^p\right) ds\right) \\ &\quad + \sum_{\substack{u < \xi_t \\ S_u > T_\delta \vee T_\varepsilon}} C \exp\left(\int_{T_\delta \vee T_\varepsilon}^{S_u} \left((1 + \delta)\frac{1}{2}g'(s)^2 - \beta(1 - \varepsilon)g(s)^p\right) ds\right). \end{aligned}$$

The first sum in the expression above is finite  $\tilde{\mathbb{Q}}_g^x$ -almost surely. To deal with the second sum, we note that, under  $\tilde{\mathbb{Q}}_g^x$ , the births along the spine are an inhomogeneous Poisson process with rate  $2\beta|\xi_t|^p$  at time  $t$ , and so the total number of offspring to time  $t$  is Poisson distributed with expectation  $2\beta \int_0^t |\xi_s|^p ds$ , conditional on  $\mathcal{G}_\infty$ . In view of Eq. (5) this is almost surely  $o(\int_0^t (1+\varepsilon)|g(s)|^p ds)$  as  $t \rightarrow \infty$ . However, as we saw above, the individual terms in the second sum decay like  $\exp(-c \int_0^t |g(s)|^p ds)$  for some constant  $c > 0$ .

Thus in all cases  $p \in [0, 2]$ , the decaying size of the summands dominates the growth in the number of summands and the Law of Large Numbers gives  $\limsup_{t \rightarrow \infty} \tilde{\mathbb{Q}}_g^x(M_g(t)|\tilde{\mathcal{G}}_\infty) < +\infty$ ,  $\tilde{\mathbb{Q}}_g^x$ -almost surely.

Using Fatou's lemma we have

$$\begin{aligned} \tilde{\mathbb{Q}}_g^x\left(\liminf_{t \rightarrow \infty} M_g(t)|\tilde{\mathcal{G}}_\infty\right) &\leq \liminf_{t \rightarrow \infty} \tilde{\mathbb{Q}}_g^x(M_g(t)|\tilde{\mathcal{G}}_\infty) \\ &\leq \limsup_{t \rightarrow \infty} \tilde{\mathbb{Q}}_g^x(M_g(t)|\tilde{\mathcal{G}}_\infty) < +\infty, \end{aligned}$$

$\tilde{\mathbb{Q}}_g^x$ -almost surely, which implies that  $\liminf_{t \rightarrow \infty} M_g(t) < +\infty$ ,  $\tilde{\mathbb{Q}}_g^x$ -almost surely. Furthermore,  $\liminf_{t \rightarrow \infty} M_g(t)$  is  $\mathcal{F}_\infty$ -measurable, and so,  $\mathbb{Q}_g^x$ -almost surely, we have  $\liminf_{t \rightarrow \infty} M_g(t) < +\infty$ .

In light of (2),  $1/M_g(t)$  is a positive  $\mathbb{Q}_g^x$ -supermartingale, which converges almost surely, whence  $M_g(t)$  converges  $\mathbb{Q}_g^x$ -almost surely. Combining these last few observations we have  $\limsup_{t \rightarrow \infty} M_g(t) = \liminf_{t \rightarrow \infty} M_g(t) < +\infty$ ,  $\mathbb{Q}_g^x$ -almost surely. Using Eq. (3) with Scheffé's lemma shows that  $M_g$  is  $\mathcal{L}^1(P^x)$ -convergent for the paths  $g$  required in Theorem 2, hence  $M_g$  is uniformly integrable.

It remains to show that  $P^x(M_g(\infty) = 0) = 0$ . For this we use the following lemma, whose proof we give separately at the end of this section.

**Lemma 3.** *Let  $q: \mathbb{R} \rightarrow [0, 1]$  be such that  $M_t := \prod_{u \in N_t} q(Y_u(t))$  is a  $P$ -martingale. Then  $q(x) \equiv q \in \{0, 1\}$ .*

If  $p(x) := P^x(M_g(\infty) = 0)$ , then applying the branching Markov property we obtain, for  $t > 0$ ,

$$p(x) = E^x(P^x(M_g(\infty) = 0|\mathcal{F}_t)) = E^x\left(\prod_{u \in N_t} p(Y_u(t))\right),$$

whence  $\prod_{u \in N_t} p(Y_u(t))$  is a martingale. Since  $E^x(M_g(\infty)) = M_g(0) > 0$  for the required paths  $g$  it follows from Lemma 3 that  $p(x) \equiv 0$ .  $\square$

**Proof of Theorem 2: zero-limit parts.** Since one of the particles alive at time  $t$  is the spine, we have that

$$M_g(t) \geq \exp\left(\int_0^t g'(s) d\tilde{B}_s + \int_0^t \left(\frac{1}{2}g'(s)^2 - \beta|\tilde{B}_s + g(s)|^p\right) ds\right).$$

We now restrict to our particular paths of interest again and use (5) and (6) to find a lower bound of

$$M_g(t) \geq C \exp\left(\int_{T_\delta \vee T_\varepsilon}^t \left((1-\delta)\frac{1}{2}g'(s)^2 - \beta(1+\varepsilon)g(s)^p\right) ds\right) \quad (9)$$

$\tilde{\mathbb{Q}}_g^x$ -almost surely for  $t > T_\delta \vee T_\varepsilon$ , where  $C > 0$  is some random, almost-surely finite, constant.

A case analysis very similar to that done in the proof of  $\mathcal{L}^1$  convergence above shows that this lower bound blows up when either:  $p \in [0, 2)$  and  $b > \hat{b}$ ;  $p \in [0, 2)$ ,  $b = \hat{b}$  and  $a > \hat{a}$ ; or  $p = 2$  and  $\lambda > \sqrt{2\beta}$ .

Consequently, in all these cases,  $\limsup_{t \rightarrow \infty} M_g(t) = +\infty$ ,  $\tilde{\mathbb{Q}}_g^x$ -almost surely. This also holds  $\mathbb{Q}_g^x$ -almost surely, and recalling (4) we obtain the result.  $\square$

**Proof of Lemma 3.** Since  $M_t$  is a martingale we have

$$q(x) = E^x M_t = \tilde{E}^x M_t \leq \tilde{E}^x q(\xi_t),$$

and so  $q(\xi_t)$  is a bounded submartingale, which must converge  $\tilde{P}$ -almost surely to some limit  $q_\infty$ . But  $\xi_t$  is recurrent, so for  $q(\xi_t)$  to converge it follows that  $q(x) \equiv q_\infty$  is constant. If  $q_\infty \in [0, 1)$  then  $M_t \rightarrow 0$  because  $N_t \rightarrow +\infty$ ,  $P^x$ -almost surely, and as  $M_t$  is uniformly integrable it must be the case that  $q(x) = E^x M_\infty = 0$ . So  $q_\infty \in \{0, 1\}$ , as claimed.  $\square$

#### 4. The right-most particle asymptotic

In this section we will prove results which, taken together, will yield Theorem 1.

**Proposition 4.** (a)  $(\beta|y|^p; \mathbb{R})$ -BBM,  $p \in [0, 2)$ .  $\liminf_{t \rightarrow \infty} t^{-\hat{b}} R_t \geq \hat{a}$   $P^x$ -a.s.  
 (b)  $(\beta y^2; \mathbb{R})$ -BBM.  $\liminf_{t \rightarrow \infty} t^{-1} \ln R_t \geq \sqrt{2\beta}$   $P^x$ -a.s.

**Proof.** (a) Define

$$B_a := \left\{ \exists u: \liminf_{t \rightarrow \infty} t^{-\hat{b}} Y_u(t) = a \right\}.$$

Then  $B_a \in \mathcal{F}_\infty$  and  $\mathbb{Q}_g^x(B_a) = \tilde{\mathbb{Q}}_g^x(B_a) = 1$ , because under  $\tilde{\mathbb{Q}}_g^x$  the spine is a Brownian motion following the path  $g(s) = as^{\hat{b}}$  (recall (1)). For  $a \in (0, \hat{a})$ ,  $M_g$  is uniformly integrable by Theorem 2 and hence  $P^x(M_g(\infty)) = 1$ . By definition we have

$$P^x(\mathbf{1}_{B_a} M_g(\infty)) = \mathbb{Q}_g^x(B_a) = 1,$$

and because both  $P^x(M_g(\infty) > 0) = 1$  and  $P^x(M_g(\infty)) = 1$ , it follows that  $P^x(B_a) = 1$ . This holds for all  $a \in (0, \hat{a})$  and the result follows.

(b) The proof of this part is almost identical: setting

$$B_\lambda := \left\{ \exists u: \liminf_{t \rightarrow \infty} t^{-1} \ln Y_u(t) = \lambda \right\},$$

we find that  $P^x(B_\lambda) = 1$  for all  $\lambda \in (0, \sqrt{2\beta})$ .  $\square$

To prove the required upper bounds we will need the following 0–1 laws.

**Lemma 5.** For all  $x, y \in \mathbb{R}$  and  $a, b > 0$ ,

$$P^x\left(\limsup_{t \rightarrow \infty} t^{-1} \ln R_t > y\right) \in \{0, 1\},$$

$$P^x\left(\limsup_{t \rightarrow \infty} t^{-b} R_t > a\right) \in \{0, 1\}.$$

**Proof.** Setting

$$q_1(x) = P^x\left(\limsup_{t \rightarrow \infty} t^{-1} \ln R_t \leq y\right),$$

$$q_2(x) = P^x\left(\limsup_{t \rightarrow \infty} t^{-b} R_t \leq a\right),$$

it is easy to check that  $\prod_{u \in N_t} q_1(Y_u(t))$  and  $\prod_{u \in N_t} q_2(Y_u(t))$  are martingales, and applying Lemma 3 yields the result.  $\square$

**Proposition 6.** (a)  $(\beta|y|^p; \mathbb{R})$ -BBM,  $p \in [0, 2)$ .  $\limsup_{t \rightarrow \infty} t^{-\hat{b}} R_t \leq \hat{a}$   $P^x$ -a.s.  
 (b)  $(\beta y^2; \mathbb{R})$ -BBM.  $\limsup_{t \rightarrow \infty} t^{-1} \ln R_t \leq \sqrt{2\beta}$   $P^x$ -a.s.



**Proof.** From Lemma 5 we know that the events in Proposition 6 have probability 0 or 1. Our method of proof is to suppose that the limsups are too large almost surely and then obtain a contradiction by showing that this would cause the martingale  $M_f$  to fail to converge, for a suitably chosen function  $f$ .

We suppose that, for  $p \in [0, 2)$ ,

$$P^x \left( \limsup_{t \rightarrow \infty} t^{-\hat{b}} R_t > a_0 \right) = 1 \quad (10)$$

for some  $a_0 > \hat{a}$ , and for  $p = 2$  that

$$P^x \left( \limsup_{t \rightarrow \infty} t^{-1} \ln R_t > \lambda_0 \right) = 1 \quad (11)$$

for some  $\lambda_0 > \sqrt{2\beta}$ . (Note that (10) and (11) imply that the corresponding  $\tilde{P}^x$ -probabilities are also 1.)

Now choose  $a_1 \in (\hat{a}, a_0)$  and set  $f_1(t) := a_1(1+t)^{\hat{b}} - a_1$ , and also choose  $\lambda_1 \in (\sqrt{2\beta}, \lambda_0)$  and set  $f_2(t) := e^{\lambda_1 t}$ . We will show that assumptions (10) and (11) force, respectively, the martingales  $M_{f_1}$  and  $M_{f_2}$  to fail to converge. (Note that if we can show that the probability in expression (10) is zero for any  $a > a_0$ , this will also imply that there the right-most particle cannot have a displacement that is asymptotically of order  $t^b$  for any  $b > \hat{b}$ .)

Let  $D(f_1)$  be the space-time region bounded above by the curve  $y = f_1(t)$  and below by the curve  $y = -f_1(t)$ , and define  $D(f_2)$  analogously. We will use the notation  $f$  and  $D(f)$  to talk about both of these functions or regions without mentioning the individual cases explicitly. The path of the initial particle, the spine  $\xi_t$ , is a  $\tilde{P}^x$ -Brownian motion and satisfies  $|\xi_t|/t \rightarrow 0$  almost surely. Hence there exists  $\tilde{P}^x$ -almost surely a random time  $T' < \infty$  such that  $\xi_t \in D(f)$  for all  $t > T'$ .

Next we observe that the spine will, throughout its (infinite) lifetime, spend an infinite amount of time in any interval  $[\varepsilon, 1)$  for  $\varepsilon > 0$ . During the time it spends in this interval it is giving birth to offspring at the points of an inhomogeneous Poisson process with rate bounded below by  $\beta\varepsilon^p > 0$ ; this assures us of the existence of an infinite sequence,  $\{T_n\}_{n \in \mathbb{N}}$ , of birth times along the path of the spine when it is in the interval  $[\varepsilon, 1)$ , with  $0 \leq T' < T_1 < T_2 < \dots$  and  $T_n \rightarrow \infty$ . Denote by  $u_n$  the label of the particle born at time  $T_n$ , for all  $n \in \mathbb{N}$ .

Each particle  $u_n$  gives rise to an independent  $(\beta|y|^p; \mathbb{R})$ -BBM with law  $P^{\xi(T_n)}$ . Almost surely, by hypotheses (10) and (11), each  $u_n$  has a descendant that leaves the space-time region  $D(f)$ . We wish to consider those particles  $u_n$  whose first descendant to leave the region  $D(f)$  does so by crossing the positive bounding curve. Since the breeding potential is symmetric about the origin, and the particles  $u_n$  are born in the interval  $[\varepsilon, 1)$ , there is at least probability  $\frac{1}{2}$  that the first descendant of each  $u_n$  to leave  $D(f)$  crosses the positive bounding curve. Almost surely, therefore, an infinite number of the particles  $u_n$  have a descendant which does this, and we will label this infinite subsequence of the  $u_n$  by  $\{v_n\}_{n \in \mathbb{N}}$ . For an offspring  $u$  of the initial ancestor  $\xi$  we will use the notation  $S_u$  for its birth time.

Thus we have shown that,  $\tilde{P}^x$ -almost surely, there exists a sequence of times  $\{J_n\}_{n \in \mathbb{N}}$  at which a descendant of the  $v_n$  first exits  $D(f)$ , i.e.

$$J_n := \inf \{ t > S_{v_n} : \exists w > v_n, S_w < t, Y_w(t) = f(t), |Y_w(s)| < f(s) \forall s \in [T', t) \},$$

and we have  $J_n \rightarrow \infty$  as  $n \rightarrow \infty$ . Further, denote by  $w_n \in N_{J_n}$  the ( $\tilde{P}^x$ -a.s. unique) particle satisfying  $w_n > v_n$  and  $Y_{w_n}(J_n) = f(J_n)$ .

To obtain a contradiction we consider the martingale  $M_f$  evaluated at the times  $J_n$ . Noting that  $f'' > 0$ , we have,  $\tilde{P}^x$ -almost surely,

$$\begin{aligned} M_f(J_n) &\geq \exp \left( \int_0^{J_n} f'(s) dY_{w_n}(s) - \int_0^{J_n} \left( \frac{1}{2} f'(s)^2 + \beta |Y_{w_n}(s)|^p \right) ds \right) \\ &= \exp \left( f'(J_n) Y_{w_n}(J_n) - f'(0) Y_{w_n}(0) - \int_0^{J_n} \left( f''(s) Y_{w_n}(s) + \frac{1}{2} f'(s)^2 + \beta |Y_{w_n}(s)|^p \right) ds \right) \\ &\geq \exp \left( f'(J_n) f(J_n) - f'(0)x - \int_0^{T'} \left( f''(s) \xi_s + \frac{1}{2} f'(s)^2 + \beta |\xi_s|^p \right) ds \right) \end{aligned}$$

$$\begin{aligned}
 & - \int_{T'}^{J_n} \left( f''(s)f(s) + \frac{1}{2}f'(s)^2 + \beta|f(s)|^p \right) ds \\
 & \geq C \exp \left( f'(J_n)f(J_n) - \int_0^{J_n} \left( f''(s)f(s) + \frac{1}{2}f'(s)^2 + \beta|f(s)|^p \right) ds \right) \\
 & \geq C' \exp \left( \int_0^{J_n} \left( \frac{1}{2}f'(s)^2 - \beta|f(s)|^p \right) ds \right),
 \end{aligned}$$

for some random,  $\tilde{P}^x$ -almost surely finite constants  $C$  and  $C'$  which only depend on the path of  $\xi$  up to time  $T'$ . This lower bound for  $M_f$  increases unboundedly when we substitute in either of the functions  $f_1$  or  $f_2$ , giving  $\limsup_{t \rightarrow \infty} M_f(t) = +\infty$ ,  $\tilde{P}^x$ -almost surely, which is a contradiction.

Recalling Lemma 5 we must have that the probabilities in expressions (10) and (11) are both 0, and the results follow.  $\square$

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